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On the Cauchy–Rassias stability of a generalized additive functional equation

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Abstract

Let X and Y be Banach spaces and $f : X \to Y$ an odd mapping. We solve the following generalized additive functional equation

$$rf\left(\frac{\sum_{j=1}^{d} x_j}{r}\right) + \sum_{\substack{\iota(j)=0,1 \\ \sum_{j=1}^{d}\iota(j)=l}} rf\left(\frac{\sum_{j=1}^{d}(-1)^{\iota(j)}x_j}{r}\right) = \left({}_{d-1}C_l - {}_{d-1}C_{l-1} + 1\right)\sum_{j=1}^{d} f(x_j)$$

for all $x_1, \ldots, x_d \in X$. Moreover we deal with the above functional equation in Banach modules over a C^*-algebra and obtain generalizations of the Cauchy–Rassias stability. The concept of Cauchy–Rassias stability for the linear mapping was originated from Th.M. Rassias's stability theorem that appeared in his paper: [Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978) 297–300].

Keywords: Banach space; Cauchy–Rassias stability; Generalized additive functional equation; Generalized additive mapping; Banach module over a C^*-algebra

1. Introduction

Ulam [20] raised the stability problem: “Give conditions in order for a linear mapping near an approximately linear mapping to exist.” Hyers [4] solved this stability problem in real Banach spaces. In 1978, Th.M. Rassias [13] provided a generalization of Hyers's theorem, which allows the Cauchy difference to be unbounded.Thus, he was able to prove the generalized stability of the linear mapping between Banach spaces. During the last three decades several mathematicians (Z. Gajda, C.‘Park, T. Trif, to mention just a few) obtained a number of substantial generalizations [1,5,6,12–19]. The following theorem which is called the Cauchy–Rassias stability is a generalized solution to the stability problem.

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Theorem A. *Let X and Y be real Banach spaces and $f : X \to Y$ a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. If there exist constants $\theta \geqslant 0$ and $p \geqslant 0$ with $p \neq 1$ satisfying*

$$\left\| f(x+y) - f(x) - f(y) \right\| \leqslant \theta\left(\|x\|^p + \|y\|^p\right)$$

for all $x, y \in X$, then there exists a unique $\mathbb{R}$-linear mapping $L : X \to Y$ such that

$$\left\| f(x) - L(x) \right\| \leqslant \frac{2\theta}{|2^p - 2|}\|x\|^p$$

for all $x \in X$.

On the other hand, P. Găvruta [2] generalized the Cauchy–Rassias stability: Let G be an abelian group and Y a Banach space. Let $f : G \to Y$ be a mapping. If there exists a function $\varphi : G \times G \to [0, \infty)$ satisfying

$$\tilde{\varphi}(x, y) = \sum_{j=0}^{\infty} 2^{-j}\varphi\left(2^j x, 2^j y\right) < \infty,$$
$$\left\| f(x+y) - f(x) - f(y) \right\| \leqslant \varphi(x, y)$$

for all $x, y \in G$, then there exists a unique additive mapping $L : G \to Y$ such that

$$\left\| f(x) - L(x) \right\| \leqslant \frac{1}{2}\tilde{\varphi}(x, x)$$

for all $x \in G$. C. Park applied this result to linear functional equations in Banach modules over a C^*-algebra and investigated several functional equations in [3,8–11].

Throughout this paper, we assume that r is a positive rational number and that d, l are integers with $1 < l < \frac{d}{2}$. We let k be a positive integer and m a positive integer with $m \leqslant d$ and note that ${}_dC_l := \frac{d!}{l!(d-l)!}$.

Let X and Y be Banach spaces and $f : X \to Y$ an odd mapping. The main purpose of this paper is to solve the following functional equation which is called a *generalized additive functional equation*

$$rf\left(\frac{\sum_{j=1}^{d} x_j}{r}\right) + \sum_{\substack{\iota(j)=0,1 \\ \sum_{j=1}^{d}\iota(j)=l}} rf\left(\frac{\sum_{j=1}^{d}(-1)^{\iota(j)}x_j}{r}\right) = ({}_{d-1}C_l - {}_{d-1}C_{l-1} + 1)\sum_{j=1}^{d} f(x_j) \tag{1.1}$$

for all $x_1, \ldots, x_d \in X$. The solution of the functional equation (1.1) is called a *generalized additive mapping*. In this paper, we obtain generalizations of the Cauchy–Rassias stability of the functional equation (1.1). Actually we show that for an odd mapping $f : X \to Y$, if there exist a function $\varphi : X^d \to [0, \infty)$ and a constant $\alpha \geqslant 0$ satisfying certain conditions and

$$\left\| rf\left(\frac{k}{r}x\right) - kf(x) \right\| \leqslant \alpha\varphi(\underbrace{x, \ldots, x}_{m \text{ times}}, \underbrace{0, \ldots, 0}_{d-m \text{ times}})$$

for all $x \in X$, then there exists a unique generalized additive mapping $L : X \to Y$ such that

$$\left\| f(x) - L(x) \right\| \leqslant \frac{\alpha}{k}\tilde{\varphi}(\underbrace{x, \ldots, x}_{m \text{ times}}, \underbrace{0, \ldots, 0}_{d-m \text{ times}})$$

for all $x \in X$. Furthermore, for a given $s \in \{0, 1, 2, \ldots, [\frac{k-1}{2}]\}$ when f satisfies

$$rf\left(\frac{k-2t}{r}x\right) = (k-2t)f(x)$$

for all $t \in \{0, 1, 2, \ldots, [\frac{k-1}{2}]\}$ with $t \neq s$ and all $x \in X$, we find concrete values of α.

On the other hand, we deal with a generalized additive functional equation in Banach modules over a C^*-algebra. For a unital C^*-algebra A with unitary group $\mathcal{U}(A)$ and left Banach modules X, Y over A, we consider an odd mapping $f : X \to Y$ and the following generalized additive functional equation

$$rf\left(\frac{\sum_{j=1}^{d} ux_j}{r}\right) + \sum_{\substack{\iota(j)=0,1 \\ \sum_{j=1}^{d}\iota(j)=l}} rf\left(\frac{\sum_{j=1}^{d}(-1)^{\iota(j)}ux_j}{r}\right) = ({}_{d-1}C_l - {}_{d-1}C_{l-1} + 1)\sum_{j=1}^{d} uf(x_j) \tag{1.2}$$

for all $u \in \mathcal{U}(A)$ and all $x_1, \ldots, x_d \in X$.

Our goal is to generalize the Cauchy–Rassias stability of the functional equation (1.2). In fact, C. Park [10,11] obtained the Cauchy–Rassias stability of the functional equation (1.2) for several special cases. However, our generalizations contain C. Park's results.

2. Generalized additive mappings in Banach spaces

In this section, we solve a generalized additive functional equation (1.1) in Banach spaces. In other words, we generalize the Cauchy–Rassias stability of the functional equation (1.1).

Throughout this section, let X and Y be Banach spaces. For a given mapping $f : X \to Y$, we define Df by the following:

$$Df(x_1, \ldots, x_d) := rf\left(\frac{\sum_{j=1}^{d} x_j}{r}\right) + \sum_{\substack{\iota(j)=0,1 \\ \sum_{j=1}^{d}\iota(j)=l}} rf\left(\frac{\sum_{j=1}^{d}(-1)^{\iota(j)}x_j}{r}\right) - ({}_{d-1}C_l - {}_{d-1}C_{l-1} + 1)\sum_{j=1}^{d} f(x_j)$$

for all $x_1, \ldots, x_d \in X$.

For the proof of our results, we first give a useful lemma.

Lemma 2.1. *For an odd mapping $f : X \to Y$, the following statements are equivalent*:

(a) *f is additive.*
(b) *$Df(x_1, \ldots, x_d) = 0$ for all $x_1, \ldots, x_d \in X$.*
(c) *$Df(x, y, \underbrace{0, \ldots, 0}_{d-2 \text{ times}}) = 0$ for all $x, y \in X$.*

Proof. The definition of Df follows that (a) implies (b) and (b) implies (c). The proof that (c) implies (a), being similar to that of [10, Lemma 2.1], is omitted. □

In the following proposition, we give sufficient conditions for an odd mapping to be additive.

Proposition 2.2. *Let $f : X \to Y$ be an odd mapping satisfying*

$$rf\left(\frac{k}{r}x\right) = kf(x)$$

for all $x \in X$. If there exists a function $\varphi : X^d \to [0, \infty)$ satisfying

$$\tilde{\varphi}(x_1, \ldots, x_d) := \sum_{j=0}^{\infty} \frac{r^j}{k^j}\varphi\left(\frac{k^j}{r^j}x_1, \ldots, \frac{k^j}{r^j}x_d\right) < \infty, \tag{2.1}$$

$$\left\|Df(x_1, \ldots, x_d)\right\| \leqslant \varphi(x_1, \ldots, x_d), \tag{2.2}$$

for all $x_1, \ldots, x_d \in X$, then f is additive.

Proof. For an odd mapping f satisfying $rf(\frac{k}{r}x) = kf(x)$, we get $\frac{r^n}{k^n}f(\frac{k^n}{r^n}x) = f(x)$ for all positive integer n and all $x \in X$. So by the definition of Df and (2.2), we have

$$\frac{r^n}{k^n}Df\left(\frac{k^n}{r^n}x_1, \ldots, \frac{k^n}{r^n}x_d\right) = Df(x_1, \ldots, x_d)$$

and

$$\frac{r^n}{k^n}\left\| Df\left(\frac{k^n}{r^n}x_1,\ldots,\frac{k^n}{r^n}x_d\right)\right\| \leqslant \frac{r^n}{k^n}\varphi\left(\frac{k^n}{r^n}x_1,\ldots,\frac{k^n}{r^n}x_d\right)$$

for all positive integer n and all $x_1,\ldots,x_d \in X$. From (2.1), we obtain

$$\lim_{n\to\infty}\frac{r^n}{k^n}\varphi\left(\frac{k^n}{r^n}x_1,\ldots,\frac{k^n}{r^n}x_d\right) = 0$$

and so it is straightforward to see that $Df(x_1,\ldots,x_d) = 0$ for all $x_1,\ldots,x_d \in X$. Therefore, f is additive by Lemma 2.1. □

A major goal of our work is to generalize the condition

$$rf\left(\frac{k}{r}x\right) = kf(x)$$

for all $x \in X$ in Proposition 2.2 and to obtain a generalization of the Cauchy–Rassias stability in Banach space.

Theorem 2.3. *Let $f : X \to Y$ be an odd mapping. If there exist a function $\varphi : X^d \to [0,\infty)$ satisfying* (2.1) *and* (2.2) *and a constant $\alpha \geqslant 0$ satisfying*

$$\left\| rf\left(\frac{k}{r}x\right) - kf(x)\right\| \leqslant \alpha\varphi(\underbrace{x,\ldots,x}_{m \text{ times}},\underbrace{0,\ldots,0}_{d-m \text{ times}}) \tag{2.3}$$

for all $x \in X$, then there exists a unique generalized additive mapping $L : X \to Y$ such that

$$\| f(x) - L(x)\| \leqslant \frac{\alpha}{k}\tilde{\varphi}(\underbrace{x,\ldots,x}_{m \text{ times}},\underbrace{0,\ldots,0}_{d-m \text{ times}}) \tag{2.4}$$

for all $x \in X$.

Proof. Replacing $\frac{k}{r}x$ by x and dividing by k in (2.3), we get

$$\left\| f(x) - \frac{r}{k}f\left(\frac{k}{r}x\right)\right\| \leqslant \frac{\alpha}{k}\varphi(\underbrace{x,\ldots,x}_{m \text{ times}},\underbrace{0,\ldots,0}_{d-m \text{ times}})$$

for all $x \in X$. Replacing x by $\frac{k^n}{r^n}x$ in the above inequality, we get

$$\left\| f\left(\frac{k^n}{r^n}x\right) - \frac{r}{k}f\left(\frac{k^{n+1}}{r^{n+1}}x\right)\right\| \leqslant \frac{\alpha}{k}\varphi\bigg(\underbrace{\frac{k^n}{r^n}x,\ldots,\frac{k^n}{r^n}x}_{m \text{ times}},\underbrace{0,\ldots,0}_{d-m \text{ times}}\bigg)$$

and multiplying by $\frac{r^n}{k^n}$ in the above inequality, we get

$$\left\| \frac{r^n}{k^n}f\left(\frac{k^n}{r^n}x\right) - \frac{r^{n+1}}{k^{n+1}}f\left(\frac{k^{n+1}}{r^{n+1}}x\right)\right\| \leqslant \frac{\alpha}{k}\frac{r^n}{k^n}\varphi\bigg(\underbrace{\frac{k^n}{r^n}x,\ldots,\frac{k^n}{r^n}x}_{m \text{ times}},\underbrace{0,\ldots,0}_{d-m \text{ times}}\bigg)$$

for all $x \in X$ and all positive integers n. By the above inequality, we have

$$\left\| \frac{r^q}{k^q}f\left(\frac{k^q}{r^q}x\right) - \frac{r^n}{k^n}f\left(\frac{k^n}{r^n}x\right)\right\| \leqslant \frac{\alpha}{k}\sum_{j=q}^{n-1}\frac{r^j}{k^j}\varphi\bigg(\underbrace{\frac{k^j}{r^j}x,\ldots,\frac{k^j}{r^j}x}_{m \text{ times}},\underbrace{0,\ldots,0}_{d-m \text{ times}}\bigg)$$

for all $x \in X$ and all nonnegative integers q, n with $q < n$. This shows that the sequence $\{\frac{r^n}{k^n} f(\frac{k^n}{r^n}x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{r^n}{k^n} f(\frac{k^n}{r^n}x)\}$ converges for all $x \in X$. So we can define a mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty} \frac{r^n}{k^n} f\left(\frac{k^n}{r^n}x\right)$$

for all $x \in X$. Since $f(-x) = -f(x)$ for all $x \in X$, we have $L(-x) = L(x)$ for all $x \in X$, which means that L is an odd mapping. On the other hand, by the same reasoning as the proof of Proposition 2.2, we have

$$\left\|DL(x_1, \dots, x_d)\right\| = \lim_{n\to\infty} \frac{r^n}{k^n} \left\|Df\left(\frac{k^n}{r^n}x_1, \dots, \frac{k^n}{r^n}x_d\right)\right\| = 0$$

for all $x_1, \dots, x_d \in X$ and so L is additive by Lemma 2.1.

In order to prove that L satisfies (2.4), if we put $q = 0$ and let $n \to \infty$ in the last inequality then we obtain

$$\left\|f(x) - L(x)\right\| \leqslant \frac{\alpha}{k} \sum_{j=0}^{\infty} \frac{r^j}{k^j} \varphi\Bigl(\underbrace{\frac{k^j}{r^j}x, \dots, \frac{k^j}{r^j}x}_{m \text{ times}}, \underbrace{0, \dots, 0}_{d-m \text{ times}}\Bigr) = \frac{\alpha}{k} \tilde{\varphi}(\underbrace{x, \dots, x}_{m \text{ times}}, \underbrace{0, \dots, 0}_{d-m \text{ times}})$$

for all $x \in X$.

Now to prove the uniqueness of L, let $L' : X \to Y$ be another additive mapping satisfying (2.4). Since L and L' are additive, we have

$$\begin{aligned}
\left\|L(x) - L'(x)\right\| &= \frac{r^n}{k^n} \left\|L\left(\frac{k^n}{r^n}x\right) - L'\left(\frac{k^n}{r^n}x\right)\right\| \\
&\leqslant \frac{r^n}{k^n} \left(\left\|L\left(\frac{k^n}{r^n}x\right) - f\left(\frac{k^n}{r^n}x\right)\right\| + \left\|L'\left(\frac{k^n}{r^n}x\right) - f\left(\frac{k^n}{r^n}x\right)\right\|\right) \\
&\leqslant 2\frac{\alpha}{k} \frac{r^n}{k^n} \tilde{\varphi}\Bigl(\underbrace{\frac{k^n}{r^n}x, \dots, \frac{k^n}{r^n}x}_{m \text{ times}}, \underbrace{0, \dots, 0}_{d-m \text{ times}}\Bigr) \\
&= 2\frac{\alpha}{k} \sum_{j=0}^{\infty} \frac{r^{n+j}}{k^{n+j}} \varphi\Bigl(\underbrace{\frac{k^{n+j}}{r^{n+j}}x, \dots, \frac{k^{n+j}}{r^{n+j}}x}_{m \text{ times}}, \underbrace{0, \dots, 0}_{d-m \text{ times}}\Bigr)
\end{aligned}$$

which goes to zero as $n \to \infty$ for all $x \in X$ by (2.1). Consequently, L is a unique additive mapping satisfying (2.4), as desired. □

Here we remark that for an odd mapping $f : X \to Y$ satisfying $rf(\frac{k}{r}x) = kf(x)$ for all $x \in X$, if there exists a function $\varphi : X^d \to [0, \infty)$ satisfying (2.1) and (2.2), then we can take $\alpha = 0$ in (2.3) and so f is additive. Therefore, Theorem 2.3 is a generalization of Proposition 2.2. As an application of Theorem 2.3, here we obtain the following corollary which is a generalization of the Cauchy–Rassias stability.

Corollary 2.4. *Let $f : X \to Y$ be an odd mapping. When $r > k$ and $p > 1$, or $r < k$ and $0 < p < 1$, if there exist constants $\theta \geqslant 0$ and $\alpha \geqslant 0$ satisfying*

$$\left\|Df(x_1, \dots, x_d)\right\| \leqslant \theta \sum_{j=1}^{d} \|x_j\|^p,$$

$$\left\|rf\left(\frac{k}{r}x\right) - kf(x)\right\| \leqslant \alpha\theta m \|x\|^p$$

for all $x_1, \dots, x_d \in X$ and all $x \in X$, then there exists a unique generalized additive mapping $L : X \to Y$ such that

$$\left\|f(x) - L(x)\right\| \leqslant \frac{\alpha\theta m}{k} \frac{r^{p-1}}{r^{p-1} - k^{p-1}} \|x\|^p$$

for all $x \in X$.

Proof. Let $\varphi : X^d \to [0,\infty)$ be $\varphi(x_1,\dots,x_d) = \theta\sum_{j=1}^d \|x_j\|^p$. When $r > k$ and $p > 1$, or $r < k$ and $0 < p < 1$, we get

$$\tilde{\varphi}(x_1,\dots,x_d) := \sum_{j=0}^{\infty} \frac{r^j}{k^j}\varphi\left(\frac{k^j}{r^j}x_1,\dots,\frac{k^j}{r^j}x_d\right) = \sum_{j=0}^{\infty}\frac{k^{j(p-1)}}{r^{j(p-1)}}\theta\sum_{i=1}^d \|x_i\|^p = \frac{\theta}{1-(\frac{k}{r})^{p-1}}\sum_{j=1}^d \|x_j\|^p.$$

By applying Theorem 2.3, there exists a unique additive mapping $L : X \to Y$ such that

$$\left\|f(x) - L(x)\right\| \leqslant \frac{\alpha}{k}\tilde{\varphi}(\underbrace{x,\dots,x}_{m \text{ times}}, \underbrace{0,\dots,0}_{d-m \text{ times}}) \leqslant \frac{\alpha\theta m}{k}\frac{r^{p-1}}{r^{p-1}-k^{p-1}}\|x\|^p$$

for all $x \in X$. $\square$

Here we apply Theorem 2.3 and also obtain a generalization of the Cauchy–Rassias stability in Banach space. In the rest of this paper, for a real number x we notice that $[x]$ is the greatest integer less than or equal to x.

Lemma 2.5. *Let $f : X \to Y$ be an odd mapping. Under the definition of Df, we have*

$$Df(\underbrace{x,\dots,x}_{k \text{ times}}, \underbrace{0,\dots,0}_{d-k \text{ times}}) = ({}_{d-k}C_l - {}_{d-k}C_{l-k} + 1)\left(rf\left(\frac{kx}{r}\right) - kf(x)\right) + \sum_{t=1}^{[\frac{k-1}{2}]} {}_kC_t({}_{d-k}C_{l-t} - {}_{d-k}C_{l+t-k})\left(rf\left(\frac{k-2t}{r}x\right) - (k-2t)f(x)\right)$$

for all $x \in X$.

Proof. Since f is an odd mapping, we note that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$. When $x_1 = x_2 = \dots = x_k = x$, $x_{k+1} = \dots = x_d = 0$, we have

$$\sum_{\substack{\iota(j)=0,1\\ \sum_{j=1}^d \iota(j)=l}} rf\left(\frac{\sum_{j=1}^d (-1)^{\iota(j)}x_j}{r}\right) = \sum_{t=1}^{k} {}_kC_t\, {}_{d-k}C_{l-t} rf\left(\frac{(k-2t)x}{r}\right)$$

$$= {}_{d-k}C_l rf\left(\frac{kx}{r}\right) - {}_{d-k}C_{l-k} rf\left(\frac{kx}{r}\right) + \sum_{t=1}^{k-1} {}_kC_t\, {}_{d-k}C_{l-t} rf\left(\frac{(k-2t)x}{r}\right)$$

for all $x \in X$. Since ${}_kC_t = {}_kC_{k-t}$ and $f(\frac{(2t-k)x}{r}) = -f(\frac{(k-2t)x}{r})$, we have

$$\sum_{t=1}^{k-1} {}_kC_t\, {}_{d-k}C_{l-t} rf\left(\frac{(k-2t)x}{r}\right) = \sum_{t=1}^{[\frac{k-1}{2}]} {}_kC_t({}_{d-k}C_{l-t} - {}_{d-k}C_{l+t-k}) rf\left(\frac{(k-2t)x}{r}\right).$$

Thus by putting $x_1 = x_2 = \dots = x_k = x$, $x_{k+1} = \dots = x_d = 0$ in the definition of Df, we obtain

$$\begin{aligned}Df(\underbrace{x,\dots,x}_{k \text{ times}}, \underbrace{0,\dots,0}_{d-k \text{ times}}) &= rf\left(\frac{kx}{r}\right) + \sum_{t=1}^{k} {}_kC_t\, {}_{d-k}C_{l-t} rf\left(\frac{(k-2t)x}{r}\right) - ({}_{d-1}C_l - {}_{d-1}C_{l-1} + 1)kf(x)\\ &= ({}_{d-k}C_l - {}_{d-k}C_{l-k} + 1)\left(rf\left(\frac{kx}{r}\right) - kf(x)\right)\\ &\quad + \sum_{t=1}^{[\frac{k-1}{2}]} {}_kC_t({}_{d-k}C_{l-t} - {}_{d-k}C_{l+t-k})\left(rf\left(\frac{(k-2t)x}{r}\right) - (k-2t)f(x)\right)\end{aligned}$$

for all $x \in X$. $\square$

Theorem 2.6. *Let $f : X \to Y$ be an odd mapping satisfying*

$$rf\left(\frac{k-2t}{r}x\right) = (k-2t)f(x)$$

for all $t \in \{1, 2, \ldots, [\frac{k-1}{2}]\}$ and all $x \in X$. If there exists a function $\varphi : X^d \to [0, \infty)$ satisfying (2.1) *and* (2.2)*, then there exists a unique generalized additive mapping $L : X \to Y$ such that*

$$\left\| f(x) - L(x) \right\| \leqslant \frac{1}{k({}_{d-k}C_l - {}_{d-k}C_{l-k} + 1)} \tilde{\varphi}(\underbrace{x, \ldots, x}_{k \text{ times}}, \underbrace{0, \ldots, 0}_{d-k \text{ times}})$$

for all $x \in X$.

Proof. Since an odd mapping $f : X \to Y$ satisfies

$$rf\left(\frac{k-2t}{r}x\right) = (k-2t)f(x)$$

for all $t \in \{1, 2, \ldots, [\frac{k-1}{2}]\}$ and all $x \in X$, by Lemma 2.5 we get

$$Df(\underbrace{x, \ldots, x}_{k \text{ times}}, \underbrace{0, \ldots, 0}_{d-k \text{ times}}) = ({}_{d-k}C_l - {}_{d-k}C_{l-k} + 1)\left(rf\left(\frac{kx}{r}\right) - kf(x)\right)$$

for all $x \in X$. So if there is a function $\varphi : X^d \to [0, \infty)$ satisfying (2.1) and (2.2), then we have $\alpha^{-1} = {}_{d-k}C_l - {}_{d-k}C_{l-k} + 1$ and $m = k$ in (2.3). Thus by Theorem 2.3, we complete the proof. □

Before closing this section, we give another generalization of the Cauchy–Rassias stability of a generalized additive functional equation (1.1).

Theorem 2.7. *Let $s \in \{1, 2, \ldots, [\frac{k-1}{2}]\}$ and let $f : X \to Y$ be an odd mapping satisfying*

$$rf\left(\frac{k-2t}{r}x\right) = (k-2t)f(x)$$

for all $t \in \{0, 1, 2, \ldots, [\frac{k-1}{2}]\}$ with $t \neq s$ and all $x \in X$. If there exists a function $\varphi : X^d \to [0, \infty)$ satisfying

$$\tilde{\varphi}(x_1, \ldots, x_d) := \sum_{j=0}^{\infty} \frac{r^j}{(k-2s)^j} \varphi\left(\frac{(k-2s)^j}{r^j}x_1, \ldots, \frac{(k-2s)^j}{r^j}x_d\right) < \infty, \tag{2.5}$$

for all $x_1, \ldots, x_d \in X$ and satisfying (2.2)*, then there exists a unique generalized additive mapping $L : X \to Y$ such that*

$$\left\| f(x) - L(x) \right\| \leqslant \frac{1}{(k-2s)_k C_s({}_{d-k}C_{l-s} - {}_{d-k}C_{l+s-k})} \tilde{\varphi}(\underbrace{x, \ldots, x}_{k \text{ times}}, \underbrace{0, \ldots, 0}_{d-k \text{ times}})$$

for all $x \in X$.

Proof. For a given $s \in \{1, 2, \ldots, [\frac{k-1}{2}]\}$, since an odd mapping $f : X \to Y$ satisfies

$$rf\left(\frac{k-2t}{r}x\right) = (k-2t)f(x)$$

for all $t \in \{0, 1, 2, \ldots, [\frac{k-1}{2}]\}$ with $t \neq s$ and all $x \in X$, Lemma 2.5 follows the equation

$$Df(\underbrace{x, \ldots, x}_{k \text{ times}}, \underbrace{0, \ldots, 0}_{d-k \text{ times}}) = {}_kC_s({}_{d-k}C_{l-s} - {}_{d-k}C_{l+s-k})\left(rf\left(\frac{k-2s}{r}x\right) - (k-2s)f(x)\right)$$

for all $x \in X$. So if there exists a function $\varphi : X^d \to [0, \infty)$ satisfying (2.5) and (2.2), then we have $\alpha^{-1} = {}_kC_s({}_{d-k}C_{l-s} - {}_{d-k}C_{l+s-k})$ and $m = k$ in (2.3). Thus by Theorem 2.3, we complete the proof. □

3. Generalized additive mappings in Banach modules over a C^*-algebra

In this section, we deal with a generalized additive functional equation (1.2) and generalize the Cauchy–Rassias stability in Banach modules over a unital C^*-algebra.

Throughout this section, let X and Y be left Banach modules over a unital C^*-algebra A with unitary group $\mathcal{U}(A)$. For a given mapping $f : X \to Y$ and $u \in \mathcal{U}(A)$, we define $D_u f$ by the following:

$$D_u f(x_1, \dots, x_d) := rf\left(\frac{\sum_{j=1}^d ux_j}{r}\right) + \sum_{\substack{\iota(j)=0,1\\ \sum_{j=1}^d \iota(j)=l}} rf\left(\frac{\sum_{j=1}^d (-1)^{\iota(j)} ux_j}{r}\right) - ({}_{d-1}C_l - {}_{d-1}C_{l-1} + 1)\sum_{j=1}^d uf(x_j)$$

for all $x_1, \dots, x_d \in X$.

At first, we need the following lemma which is actually contained in the proof of [11, Theorem 3.1] and here we give an elementary proof.

Lemma 3.1. *For an odd mapping $f : X \to Y$, the following statements are equivalent*:

(a) *f is A-linear additive.*
(b) *$D_u f(x_1, \dots, x_d) = 0$ for all $u \in \mathcal{U}(A)$ and all $x_1, \dots, x_d \in X$.*
(c) *$D_u f(x, y, \underbrace{0, \dots, 0}_{d-2 \text{ times}}) = 0$ for all $u \in \mathcal{U}(A)$ and all $x, y \in X$.*
(d) *f is additive and $D_u f(x, \underbrace{0, \dots, 0}_{d-1 \text{ times}}) = 0$ for all $u \in \mathcal{U}(A)$ and all $x \in X$.*

Proof. By the definition of $D_u f$, (a) implies (b) and (b) implies (c). If we put $u = 1$ in $D_u f$ then by Lemma 2.1, f is additive and so (c) implies (d). In order to prove that (d) implies (a), we consider the following equation

$$D_u f(x, \underbrace{0, \dots, 0}_{d-1 \text{ times}}) = ({}_{d-1}C_l - {}_{d-1}C_{l-1} + 1)\left(rf\left(\frac{ux}{r}\right) - uf(x)\right)$$

for all $u \in \mathcal{U}(A)$ and all $x \in X$. The assumption

$$D_u f(x, \underbrace{0, \dots, 0}_{d-1 \text{ times}}) = 0$$

implies $f(ux) = uf(x)$ for all $u \in \mathcal{U}(A)$ and all $x \in X$. Now we consider an unitary $\mu \in \mathbb{T}^1$ where $\mathbb{T}^1 := \{\mu \in \mathbb{C} \mid |\mu| = 1\}$. Since $f(\mu x) = \mu f(x)$ for all $\mu \in \mathbb{T}^1$, f is $\mathbb{C}$-linear by [7, Lemma 2.1]. As well known, since any element a in a C^*-algebra A is the form $a = \sum_{i=1}^n \lambda_i u_i$ for $\lambda_i \in \mathbb{C}$ and $u_i \in \mathcal{U}(A)$, we have

$$f(ax) = f\left(\sum_{i=1}^n \lambda_i u_i x\right) = \sum_{i=1}^n \lambda_i f(u_i x) = \sum_{i=1}^n \lambda_i u_i f(x) = af(x),$$

which implies that

$$f(ax + by) = af(x) + bf(y)$$

for all $a, b \in A$ and all $x, y \in X$. Thus f is A-linear additive. □

In the following proposition, we give sufficient conditions for an odd mapping to be A-linear additive.

Proposition 3.2. *Let $f : X \to Y$ be an odd mapping satisfying*

$$rf\left(\frac{k}{r}x\right) = kf(x)$$

for all $x \in X$. If there exists a function $\varphi : X^d \to [0, \infty)$ satisfying

$$\tilde{\varphi}(x_1, \ldots, x_d) := \sum_{j=0}^{\infty} \frac{r^j}{k^j} \varphi\left(\frac{k^j}{r^j}x_1, \ldots, \frac{k^j}{r^j}x_d\right) < \infty, \tag{3.1}$$

$$\left\| D_u f(x_1, \ldots, x_d) \right\| \leqslant \varphi(x_1, \ldots, x_d) \tag{3.2}$$

for all $u \in \mathcal{U}(A)$ and all $x_1, \ldots, x_d \in X$, then f is A-linear additive.

Proof. By proceeding like the proof of Proposition 2.2, it is not difficult to show that $D_u f(x_1, \ldots, x_d) = 0$ for all $u \in \mathcal{U}(A)$ and all $x_1, \ldots, x_d \in X$ and so f is A-linear additive by Lemma 3.1. □

In the following theorem, we generalize the condition

$$rf\left(\frac{k}{r}x\right) = kf(x)$$

for all $x \in X$ in Proposition 3.3 and obtain a generalization of the Cauchy–Rassias stability in Banach modules over a unital C^*-algebra.

Theorem 3.3. *Let $f : X \to Y$ be an odd mapping. If there exist a function $\varphi : X^d \to [0, \infty)$ satisfying* (3.1) *and* (3.2) *and a constant $\alpha \geqslant 0$ satisfying*

$$\left\| rf\left(\frac{k}{r}x\right) - kf(x) \right\| \leqslant \alpha\varphi(\underbrace{x, \ldots, x}_{m \text{ times}}, \underbrace{0, \ldots, 0}_{d-m \text{ times}}) \tag{3.3}$$

for all $x \in X$, then there exists a unique A-linear generalized additive mapping $L : X \to Y$ such that

$$\left\| f(x) - L(x) \right\| \leqslant \frac{\alpha}{k} \tilde{\varphi}(\underbrace{x, \ldots, x}_{m \text{ times}}, \underbrace{0, \ldots, 0}_{d-m \text{ times}}) \tag{3.4}$$

for all $x \in X$.

Proof. By the same reasoning as the proof of Theorem 2.3, there exists a unique additive mapping $L : X \to Y$ defined by

$$L(x) := \lim_{n\to\infty} \frac{r^n}{k^n} f\left(\frac{k^n}{r^n}x\right)$$

for all $x \in X$.

Thus by (3.1) and (3.2), we obtain

$$\left\| D_u L(x_1, \ldots, x_d) \right\| = \lim_{n\to\infty} \frac{r^n}{k^n} \left\| D_u f\left(\frac{k^n}{r^n}x_1, \ldots, \frac{k^n}{r^n}x_d\right) \right\| \leqslant \lim_{n\to\infty} \frac{r^n}{k^n} \varphi\left(\frac{k^n}{r^n}x_1, \ldots, \frac{k^n}{r^n}x_d\right) = 0$$

for all $u \in \mathcal{U}(A)$ and all $x_1, \ldots, x_d \in X$. Therefore we conclude that $D_u L(x_1, \ldots, x_d) = 0$ for all $u \in \mathcal{U}(A)$ and all $x_1, \ldots, x_d \in X$ and so L is A-linear additive by Lemma 3.1. □

The following corollary is a generalization of the Cauchy–Rassias stability which contains several results in [10, 11]. By applying Theorem 3.3 since the proof of the following corollary is similar to that of Corollary 2.4, we omit the details.

Corollary 3.4. *Let $f : X \to Y$ be an odd mapping. When $r > k$ and $p > 1$, or $r < k$ and $0 < p < 1$, if there exist constants $\theta \geqslant 0$ and $\alpha \geqslant 0$ satisfying*

$$\left\| D_u f(x_1, \dots, x_d) \right\| \leqslant \theta \sum_{j=1}^{d} \|x_j\|^p,$$

$$\left\| rf\left(\frac{k}{r}x\right) - kf(x) \right\| \leqslant \alpha\theta m \|x\|^p$$

for all $u \in \mathcal{U}(A)$, all $x_1, \dots, x_d \in X$, and all $x \in X$, then there exists a unique A-linear generalized additive mapping $L : X \to Y$ such that

$$\left\| f(x) - L(x) \right\| \leqslant \frac{\alpha\theta m}{k} \frac{r^{p-1}}{r^{p-1} - k^{p-1}} \|x\|^p$$

for all $x \in X$.

Now we apply Theorem 3.3 and obtain generalizations of the Cauchy–Rassias stability of a generalized additive functional equation in Banach modules over a C^*-algebra. At first, we need the following lemma.

Lemma 3.5. *Let $f : X \to Y$ be an odd mapping. Under the definition of $D_u f$, we have*

$$\begin{aligned} D_u f(\underbrace{x, \dots, x}_{k \text{ times}}, \underbrace{0, \dots, 0}_{d-k \text{ times}}) &= ({}_{d-k}C_l - {}_{d-k}C_{l-k} + 1)\left(rf\left(\frac{kux}{r}\right) - kuf(x) \right) \\ &\quad + \sum_{t=1}^{[\frac{k-1}{2}]} {}_kC_t({}_{d-k}C_{l-t} - {}_{d-k}C_{l+t-k})\left(rf\left(\frac{k-2t}{r}ux\right) - (k-2t)uf(x) \right) \end{aligned}$$

for all $u \in \mathcal{U}(A)$ and all $x \in X$.

Proof. By the same reasoning of Lemma 2.5 and the definition of $D_u f$, we obtain the desired equation. □

The following result is a generalization of the Cauchy–Rassias stability of a generalized additive functional equation.

Theorem 3.6. *Let $f : X \to Y$ be an odd mapping satisfying*

$$rf\left(\frac{k-2t}{r}x\right) = (k-2t)f(x)$$

for all $t \in \{1, 2, \dots, [\frac{k-1}{2}]\}$ and all $x \in X$. If there is a function $\varphi : X^d \to [0, \infty)$ satisfying (3.1) *and* (3.2)*, then there exists a unique A-linear generalized additive mapping $L : X \to Y$ such that*

$$\left\| f(x) - L(x) \right\| \leqslant \frac{1}{k({}_{d-k}C_l - {}_{d-k}C_{l-k} + 1)} \tilde{\varphi}(\underbrace{x, \dots, x}_{k \text{ times}}, \underbrace{0, \dots, 0}_{d-k \text{ times}})$$

for all $x \in X$.

Proof. Since an odd mapping $f : X \to Y$ satisfies

$$rf\left(\frac{k-2t}{r}x\right) = (k-2t)f(x)$$

for all $t \in \{1, 2, \dots, [\frac{k-1}{2}]\}$ and all $x \in X$, by Lemma 3.5 we get

$$D_1 f(\underbrace{x, \dots, x}_{k \text{ times}}, \underbrace{0, \dots, 0}_{d-k \text{ times}}) = ({}_{d-k}C_l - {}_{d-k}C_{l-k} + 1)\left(rf\left(\frac{kx}{r}\right) - kf(x) \right)$$

for all $x \in X$. So, if there is a function $\varphi : X^d \to [0,\infty)$ satisfying (3.1) and (3.2), then we have $\alpha^{-1} = {}_{d-k}C_l - {}_{d-k}C_{l-k} + 1$ and $m = k$ in (3.3). Thus by Theorem 3.3, we complete the proof. □

Remark 3.7. When $k = 2$, the condition (3.3) is satisfied automatically and so Theorem 3.6 is the generalization of [11, Theorem 3.1]. Actually if we let $k = 2$ in Corollary 3.6, then there exists a unique A-linear generalized additive mapping $L : X \to Y$ such that

$$\left\| f(x) - L(x) \right\| \leqslant \frac{1}{2({}_{d-2}C_l - {}_{d-2}C_{l-2} + 1)} \tilde{\varphi}(x, x, \underbrace{0, \ldots, 0}_{d-2 \text{ times}})$$

for all $x \in X$. Moreover, when $r = 1$ and $k = 3$ in Theorem 3.6, we obtain [10, Theorem 3.1].

Finally we introduce the following theorem in order to complete our generalizations of the stability of a generalized additive functional equation. The proof of the next theorem, being similar to that of Theorem 3.6, is omitted.

Theorem 3.8. *Let $s \in \{1, 2, \ldots, [\frac{k-1}{2}]\}$ and let $f : X \to Y$ be an odd mapping satisfying*

$$rf\left(\frac{k-2t}{r}x\right) = (k-2t)f(x)$$

for all $t \in \{0, 1, 2, \ldots, [\frac{k-1}{2}]\}$ with $t \neq s$ and all $x \in X$. If there exists a function $\varphi : X^d \to [0,\infty)$ satisfying

$$\tilde{\varphi}(x_1, \ldots, x_d) := \sum_{j=0}^{\infty} \frac{r^j}{(k-2s)^j} \varphi\left(\frac{(k-2s)^j}{r^j} x_1, \ldots, \frac{(k-2s)^j}{r^j} x_d\right) < \infty \tag{3.5}$$

for all $x_1, \ldots, x_d \in X$ and satisfying (3.2), then there exists a unique A-linear generalized additive mapping $L : X \to Y$ such that

$$\left\| f(x) - L(x) \right\| \leqslant \frac{1}{(k-2s)_k C_s ({}_{d-k}C_{l-s} - {}_{d-k}C_{l+s-k})} \tilde{\varphi}(\underbrace{x, \ldots, x}_{k \text{ times}}, \underbrace{0, \ldots, 0}_{d-k \text{ times}})$$

for all $x \in X$.

Remark 3.9. When $r = 3$ and $k = 3$, since $s = 1$ and $t = 0$ in Theorem 3.8 there exists a unique A-linear generalized additive mapping $L : X \to Y$ such that

$$\left\| f(x) - L(x) \right\| \leqslant \frac{1}{3({}_{d-3}C_{l-1} - {}_{d-3}C_{l-2})} \tilde{\varphi}(x, x, x, \underbrace{0, \ldots, 0}_{d-3 \text{ times}})$$

for all $x \in X$, which implies that Theorem 3.8 is a generalization of [10, Theorem 3.7].

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